Identities inside the Gluon and the Graviton Scattering Amplitudes— an approach to BCJ conjecture

The duality between the color and kinematic factors, gluon and graviton scattering amplitude via Heterotic string theory

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based on arXiv:1003.1732, Henry Tye and Y. Z.

BCJ conjecture

M-gluon tree amplitude in pure Yang-Mills theory is

$$\mathcal{A}_M^{\text{YM}} = \sum_i^{(2M-5)!!} \frac{c_i n_i}{P_i}.$$ c_i color factor. n_i kinematic factors. P_i poles.

Z.Bern, J.Carrasco and H.Johansson conjecture that, which is checked by computer until $M \leq 8$. (hep-ph/0805.3993)

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1. If three color factors satisfy (Jacobi) $c_i + c_j + c_k = 0$, then the corresponding $n_i + n_j + n_k = 0$.
2. M-graviton tree amplitude in Einstein theory is

$$A_M^{\mathrm{Grav}} = \sum_{i=1}^{(2M-5)!!} \frac{n_i n_i}{P_i}.$$ same n_i and P_i

Checked up to $M = 8$, with computer.

4-gluon example

Scattering amplitude for four gluons, (k_1, a_1, ζ_1), (k_2, a_2, ζ_2), (k_3, a_3, ζ_3) and (k_4, a_4, ζ_4) is easily obtained by Feynman rules,

$$\mathcal{A}_4^{\mathrm{YM}} = \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}$$

where the 4-point vertex contribution is absorb into s, t and u channels. $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$, $c_t = f^{a_2 a_3 b} f^{b a_1 a_4}$ and $c_u = f^{a_3 a_1 b} f^{b a_2 a_4}$.

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$$\begin{aligned} n_s &= i[(\zeta_1 \cdot \zeta_2)(k_2 - k_1) - (2k_2 \cdot \zeta_1)\zeta_2 + (2k_1 \cdot \zeta_2)\zeta_1] \\ &\quad \times[(\zeta_3 \cdot \zeta_4)(k_4 - k_3) - (2k_4 \cdot \zeta_3)\zeta_4 + (2k_3 \cdot \zeta_4)\zeta_3] \\ &\quad -i[(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot \zeta_4) - (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot \zeta_3)]s \\ &\quad n_t = \ldots, n_u = \ldots \end{aligned}$$

4-gluon scattering example

It is easy to see that, by Jacobi identity,

$$c_s + c_t + c_u = f^{a_1 a_2 b} f^{b a_3 a_4} + f^{a_2 a_3 b} f^{b a_1 a_4} + f^{a_3 a_1 b} f^{b a_2 a_4} = 0$$

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However, it is amazing that the kinematic factors satisfy the same identity as the color factors,

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Why do the color factors and the kinematic factor satisfy the same kind of identity?

5-gluon scattering example

More complicated, 15 channels

$$A_5^{\rm YM} = \frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{15}s_{23}} + \frac{c_3 n_3}{s_{12}s_{34}} + \frac{c_4 n_4}{s_{23}s_{45}} + \frac{c_5 n_5}{s_{15}s_{34}} + \frac{c_6 n_6}{s_{14}s_{25}} + \frac{c_7 n_7}{s_{14}s_{23}} + \frac{c_8 n_8}{s_{34}s_{25}} + \frac{c_9 n_9}{s_{13}s_{25}} + \frac{c_{10} n_{10}}{s_{13}s_{24}} + \frac{c_{11} n_{11}}{s_{15}s_{24}} + \frac{c_{12} n_{12}}{s_{12}s_{35}} + \frac{c_{13} n_{13}}{s_{24}s_{35}} + \frac{c_{14} n_{14}}{s_{14}s_{35}} + \frac{c_{15} n_{15}}{s_{13}s_{45}}$$

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Still, the color factors and the kinematic factors satisfy the same identities,

$$\begin{aligned} c_4 + c_{15} - c_1 &= 0, & n_4 + n_{15} - n_1 &= 0 \\ c_4 + c_7 - c_2 &= 0, & n_4 + n_7 - n_2 &= 0 \\ c_8 + c_9 - c_6 &= 0, & n_8 + n_9 - n_6 &= 0 \\ c_3 + c_8 - c_5 &= 0, & n_3 + n_8 - n_5 &= 0 \\ &\ldots \end{aligned}$$

10 identities for c_i's, and 10 dual identities for n_i's.

M-gluon scattering and M-graviton scattering

More and more channels for the growing M. The number of channels, color factors, kinematic factors are all $(2M-5)!!$. For $M \leq 8$, BCJ shows that if $c_i + c_j + c_k = 0$, then $n_i + n_j + n_k = 0$ and for tree level M-graviton scattering amplitude, $M \leq 8$,

$$A_M^{\text{Grav}} = \sum_{i=1}^{(2M-5)!!} \frac{n_i n_i}{P_i}, \quad A_4^{\text{Grav}} = \frac{n_s n_s}{s} + \frac{n_u n_u}{u} + \frac{n_t n_t}{t}$$

Question: what is the original of these dualities?

$$c_i + c_j + c_k = 0 \Leftrightarrow n_i + n_j + n_k = 0$$
$$A_M^{\text{Gluon}} = \sum \frac{c_i n_i}{P_i} \Leftrightarrow A_M^{\text{Grav}} = \sum \frac{n_i n_i}{P_i}$$

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Elegant answer: Heterotic string theory.

As a closed string theory,

$$\text{State} = \text{left-moving sector} \times \text{right-moving sector}$$

Massless left-moving sector

1. Vector sector. $i\xi_\mu \partial X^\mu e^{ik_\nu X^\nu}$
2. Color sector. $e^{ik_\nu X^\nu + iK_I X^I}$ or $i\zeta_I \partial X^I e^{ik_\nu X^\nu}$.

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$$\begin{aligned} \text{Gluon} &= \text{color sector} \times \text{vector sector} \\ \text{Graviton} &= \text{vector sector} \times \text{vector sector}\big|_{\xi_\mu \zeta_\nu \to \epsilon_{\mu\nu}} \\ \text{Gluino} &= \text{color sector} \times \text{spinor sector} \\ \text{Gravitino} &= \text{vector sector} \times \text{spinor sector} \end{aligned}$$

KLT

In the the low energy limit, $\alpha' \to 0$, the string scattering amplitude will reduce to the Yang-Mills or gravitity scattering amplitudes.
Closed string amplitude can be written as the product of open string amplitudes, KLT relation, (H.Kawai, D.C.Lewellen and H.Tye),

$$\begin{aligned} & \text{closed string amplitude} \\ = & \sum(...)(\text{left open string amplitude}) \times (\text{right open string amplitude}) \end{aligned}$$

The analytic property of the left-moving open amplitude will give the Jacobi identity while the same kind of analytic property of the right-moving amplitude will give the BCJ dual identities. Finally, the KLT product will naturally gives the duality between the guage/gravity tree amplitude.

Left-moving open amplitude

For 4-gluon scattering, we have 3 partial amplitudes for the left-moving color sectors,

$$\mathbf{A}^{L(c)}_{2134} = co(2134)\int_{-\infty}^{0} dx_2\ (-x_2)^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2)$$

$$\mathbf{A}^{L(c)}_{1234} = co(1234)\int_{0}^{1} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2)$$

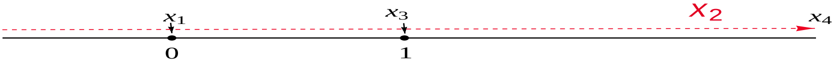
$$\mathbf{A}^{L(c)}_{1324} = co(1324)\int_{1}^{\infty} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2)$$

where $co(1234)$ and etc are the product of co-cycles, which can only be ± 1. $f(x)$ contains the possible polarization in lattice, i.e., color index in Cartan sub-algebra. The three amplitudes are related!

Analytic continuation

The relation between the partial amplitudes was known, (E. Plahte, Nuovo Cim.A66:713-733,1970). Recent derivation, by analytic continuation, (N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, Phys. Rev. Lett. 103, 161602 (2009), S. Stieberger, 0907.2211[hep-th].), or by CFT method, (R. Boels and N. Obers, 1003.1732[hep-th].)

$$\int_{-\infty}^{\infty} dx_2 \; x_2^{\cdots}(1-x_2)^{\cdots} f(x_2) = 0$$



$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathbf{A}^{L(c)}_{2134} + \mathbf{A}^{L(c)}_{1234} + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathbf{A}^{L(c)}_{1324} = 0.$$

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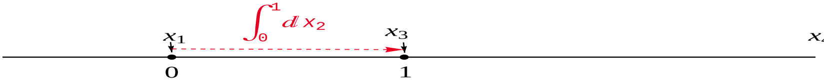

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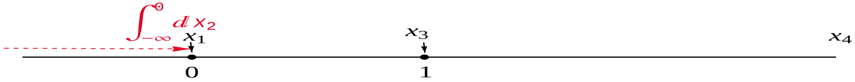

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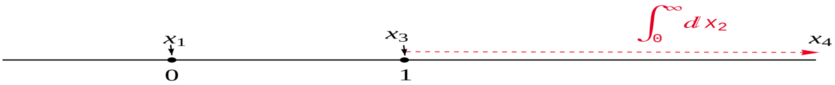

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Low energy limit

In the low energy limit, i.e., the zero slope limit only the massless state (gluon, graviton, etc.) survived so we get the field theory,

$$\lim_{\alpha' \to 0} \mathbf{A}_{1234}^{L(c)} = A_{1234}^{L(c)}$$

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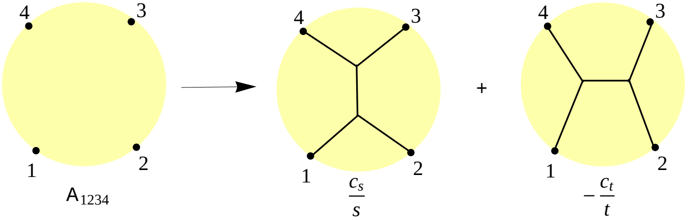
$$\lim_{\alpha' \to 0} \mathbf{A}^{L(c)}_{1234} = A^{L(c)}_{1234}$$

The contour integral identity is reduced to

$$A^{L(c)}_{2134} + A^{L(c)}_{1234} + A^{L(c)}_{1324} = 0, \text{ real part}$$
$$sA^{L(c)}_{2134} = tA^{L(c)}_{1324}, \text{ imaginary part}$$

Channels

One string amplitude, in the low energy limit, will decompose into several



channels,

$$A^{L(c)}_{2134} = -\frac{\tilde{c}_s}{s} + \frac{c_u}{u}, \; A^{L(c)}_{1234} = \frac{c_s}{s} - \frac{\tilde{c}_t}{t}, \; A^{L(c)}_{1324} = -\frac{\tilde{c}_u}{u} + \frac{c_t}{t}.$$

Plug into the contour integral identities,

$$A^{L(c)}_{2134} + A^{L(c)}_{1234} + A^{L(c)}_{1324} = 0, \quad \Rightarrow \quad \tilde{c}_s = c_s, \quad \tilde{c}_u = c_u, \quad \tilde{c}_t = c_t$$

$$sA^{L(c)}_{2134} = tA^{L(c)}_{1324}, \quad \Rightarrow \quad c_s + c_t + c_u = 0.$$

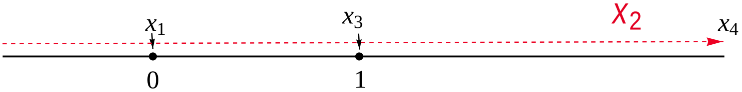
Right-moving amplitude

$$\mathbf{A}_{1234}^{R(v)} = \int_0^1 dx_2 \; x_2^{\frac{\alpha'}{2} k_1 \cdot k_2} (1 - x_2)^{\frac{\alpha'}{2} k_2 \cdot k_3} \bar{f}(x_2), \text{etc.}$$

$$\bar{f}(x_2) = \exp\left(\frac{\alpha'}{2} \sum_{i>j} \frac{\zeta_i \cdot \zeta_j}{(x_i - x_j)^2} - \frac{\alpha'}{2} \sum_{i \neq j} \frac{\zeta_i \cdot k_j}{x_i - x_j} \right) \Big|_{\text{multiple-linear}}.$$

The contour integral in x_2 gives,

$$e^{i\pi(\frac{\alpha'}{2} k_1 \cdot k_2)} \mathbf{A}_{2134}^{R(v)} + \mathbf{A}_{1234}^{R(v)} + e^{-i\pi(\frac{\alpha'}{2} k_2 \cdot k_3)} \mathbf{A}_{1324}^{R(v)} = 0.$$



kinematic identity

$$A^{R(v)}_{2134} = -\frac{n_s}{s} + \frac{n_u}{u},\ A^{R(v)}_{1234} = \frac{n_s}{s} - \frac{n_t}{t},\ A^{R(v)}_{1324} = -\frac{n_u}{u} + \frac{n_t}{t}.$$

Unlike the c_i's, the definition of n_s, n_t and n_u is not unique because we can move the contact terms between each other, $n'_s = n_s + cs$, $n'_t = n_t + ct$, $n'_u = n_u + cu$.

In the low-energy limit, the imaginary part of the contour integral identity,

$$sA^{R(v)}_{2134} = tA^{R(v)}_{1324}$$

gives,

$$n_s + n_t + n_u = 0,$$

The duality between this and $c_s + c_t + c_u = 0$ comes from the same contour. This identity is invariant under the contact term rearrangement,

$$n'_s + n'_t + n'_u = n_s + n_t + n_u + c(s+t+u) = 0$$

4-gluon and 4-graviton amplitudes

KLT, (color) $\times$ (vector)

$$\mathcal{A}^{\text{het}}_{\text{4-gluon}} \propto \sin\left(\frac{\pi\alpha' k_2 \cdot k_3}{2}\right) \times \mathbf{A}^{L(c)}_{1234}\mathbf{A}^{R(v)}_{1324}.$$

in the low energy limit,

$$\begin{aligned} &\mathcal{A}_{\text{4-gluon}} \\ &\propto t\left(\frac{c_s}{s}-\frac{c_t}{t}\right)\left(-\frac{n_u}{u}+\frac{n_t}{t}\right) \\ &= \frac{c_s n_s}{s}+\frac{c_u n_u}{u}+\frac{c_t n_t}{t}, \end{aligned}$$

We used the identities $c_s + c_t + c_u = 0$ and $n_s + n_t + n_u = 0$.

KLT, (vector) $\times$ (vector)

$$\mathcal{A}^{\text{het}}_{\text{4-graviton}} \propto \sin\left(\frac{\pi\alpha' k_2 \cdot k_3}{2}\right) \times \mathbf{A}^{L(v)}_{1234}\mathbf{A}^{R(v)}_{1324}.$$

in the low energy limit,

$$\begin{aligned} &\mathcal{A}_{\text{4-graviton}} \\ &\propto t\left(\frac{n_s}{s}-\frac{n_t}{t}\right)\left(-\frac{n_u}{u}+\frac{n_t}{t}\right) \\ &= \frac{n_s n_s}{s}+\frac{n_u n_u}{u}+\frac{n_t n_t}{t}, \end{aligned}$$

We used the identity $n_s + n_t + n_u = 0$ twice.

M-gluon

This method is easily generalized for arbitary M-gluon tree amplitude.
New feature There are many different ways to do contour integral.
New feature: One contour integral argument several color (kinematic identities). Example: x_2 contour integral in $A^{L(c)}_{12345}$,

$$-\frac{c_3+c_8-c_5}{s_{34}}-\frac{c_4-c_2+c_7}{s_{23}}+\frac{c_4+c_{15}-c_1}{s_{45}}+\frac{c_8+c_9-c_6}{s_{25}}=0.$$

Read the residues

$$c_3+c_8-c_5=0,\ c_4-c_2+c_7=0,\ c_4+c_{15}-c_1=0,\ c_8+c_9-c_6=0.$$

By detailed combinatorics, we proved that for arbitary M, the contour integral identities will give all the color identities between c_i's.

The subtlety in n_i's

There is a subtlety for $M \geq$ since n_i contains the contact terms, for example,

$$-\frac{n_3+n_8-n_5}{s_{34}}-\frac{n_4-n_2+n_7}{s_{23}}+\frac{n_4+n_{15}-n_1}{s_{45}}+\frac{n_8+n_9-n_6}{s_{25}}=0,$$

n_3, n_8 and n_5 may contain contact terms which are proportional to s_{34}. In general $n_3+n_8-n_5 \propto s_{34}$, but there is a choice such that $\tilde{n}_3+\tilde{n}_8-\tilde{n}_5=0$. We think that (still in progress),

- there exist a way to rearrange the contact terms in n_i's such that $n_i+n_j+n_k=0$ for arbitary M.
- such a way is not unique and actually these choices form a subspace with the dimension $(M-2)!-(M-3)!$.

Summary

1. (Up to the subtlety of the contact terms), a clear approach to BCJ conjecture to show the dualities between color/kinematic identities and gluon/graviton are natural.
2. The number of kinematic factors n_i dropped dramatically, so the calculation is simplifyed.
3. This method can be used for gluino and gravitinos amplitudes.

Further directions,

1. Combined with other methods? BCFW recursion, twistor spaces...
2. The loop amplitude is related to the tree amplitude via unitarity relations. So the BCJ conjecture would be generalized to the loop amplitude case.
3. KLT relation, applied in heterotic string theory, seems to give a duality between the gauge amplitude and gravity amplitude, but different from AdS/CFT. Does this relation illustrate the gauge and gravity in different regime?